ON THE CONSTRUCTION OF PERIODIC SOLUTIONS OF QUASI-LINEAR AUTONOMOUS SYSTEMS WITH SEVERAL DEGREES OF FREEDOM

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PMM Vol.26, No.2, 1962, pp. 358-364

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(Received December 2, 1961)

1. In [1] there was considered a quasi-linear system of the form

$$\sum_{k=1}^{n} (a_{ik}\ddot{x}_k + c_{ik}x_k) = \mu F_i\ (x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n, \mu) \qquad (i = 1, \ldots, n) \qquad (1.1)$$

under the assumption that the right-hand sides of the equations were analytic, and that the parameter μ was small. The generated system was a linear conservative system with constant coefficients, whose kinetic and potential energies could be expressed by means of positive definite quadratic forms. Furthermore, it was assumed that the roots of the frequency equation

$$\Delta(\omega^2) = | c_{ik} - \omega^2 a_{ik} | = 0 \qquad (1.2)$$

were all distinct.

In that work it was asserted that if the generated system contains l frequencies which are comensurate with each other, for example, $\omega_1, \ldots, \omega_l$, and which correspond to a periodic solution with some period T_0, then the corresponding periodic solution of the original nonlinear system, which reduces to the generated system when $\mu = 0$, will have a form analogous to the form of the solution of the generated system. Let us denote by $p_k^{(r)}$ the relations of the algebraic cofactors of the corresponding elements of the determinant (1.2):

$$p_k^{(r)} = \frac{\Delta_{ik}(\omega_r^2)}{\Delta_{i1}(\omega_r^2)}, \qquad p_1^{(r)} = 1 \qquad (i=1,\ldots, n) \qquad (1.3)$$

The periodic solution of the generated system with period T_0 has the form

$$x_{k0}(t) = \sum_{r=1}^{l} p_k^{(r)} \left(A_r \cos \omega_r t + \frac{B_r}{\omega_r} \sin \omega_r t \right) \qquad (k = 1, \ldots, n) \tag{1.4}$$

(one of the coefficients B_r can be assumed to be zero because the system is autonomous). It was, therefore, asserted that the solution of the system (1.1) also had the form

$$x_k(t) = \sum_{r=1}^{l} p_k^{(r)} x^{(r)}(t) \qquad (k = 1, \ldots, n)$$

But this assertion is true only when $l = n$, i.e. when all frequencies of the generated system are comensurate with each other.

2. We shall consider the case when l frequencies $\omega_1, \omega_2, \ldots, \omega_l$ of the generated system are comensurate with each other but when $l < n$. In this case, as was shown above, there exists a periodic solution of the generated system with some period T_0 corresponding to the mentioned frequencies. The solution of the generated system has the form (1.4). Since the system is autonomous, we may assume that $B_1 = 0$.

By the method of a small parameter, we will try to find a periodic solution of the original nonlinear system with period $T_0 + \alpha$, which reduces to the above mentioned solution of the generated system when $\mu = 0$. For the original system (1.1) we will take the following initial conditions*

$$\begin{aligned} x_k(0) &= \sum_{r=1}^{l} p_k^{(r)} (A_r + \beta_r) + \sum_{r=l+1}^{n} p_k^{(r)} \varphi_{r-l}(\beta_1, \ldots, \beta_l, \gamma_2, \ldots, \gamma_l, \mu) \\ & \qquad\qquad (k = 1, \ldots, n) \\ \dot{x}_k(0) &= \sum_{r=2}^{l} p_k^{(r)} (B_r + \gamma_r) + \sum_{r=l+1}^{n} p_k^{(r)} \psi_{r-l}(\beta_1, \ldots, \beta_l, \gamma_2, \ldots, \gamma_l, \mu) \end{aligned} \tag{2.1}$$

Because of the autonomicity of the system it is assumed that $\gamma_1 = 0$. The functions φ_{r-l} and ψ_{r-l} are analytical from their arguments [2], and

$$\varphi_{r-l}(\beta_1, \ldots, \beta_l, \gamma_2, \ldots, \gamma_l, 0) = 0, \qquad \psi_{r-l}(\beta_1, \ldots, \beta_l, \gamma_2, \ldots, \gamma_l, 0) = 0$$

Thus the solution of the system (1.1) can be written in the following form

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* Strictly speaking, the parameters φ and ψ are functions of $A + \beta$, $B + \gamma$ and μ. Iu.M. Kopnin was the first to call attention to this fact.

$$x_k(t, \beta, \gamma, \mu) = (A_1 + \beta_1)\cos\omega_1 t + \sum_{r=2}^{l} p_k^{(r)} \left[(A_r + \beta_r)\cos\omega_r t + \frac{B_r + \gamma_r}{\omega_r}\sin\omega_r t\right] +$$

$$+ \sum_{r=l+1}^{n} p_k^{(r)} \left[\varphi_{r-l}(\beta,\gamma,\mu)\cos\omega_r t + \frac{\psi_{r-l}(\beta,\gamma,\mu)}{\omega_r}\sin\omega_r t\right] +$$

$$+ \sum_{m=1}^{\infty} \left[C_{km}(t) + \sum_{r=1}^{l} \frac{\partial G_{km}}{\partial A_r}\beta_r + \sum_{r=2}^{l} \frac{\partial C_{km}}{\partial B_r}\gamma_r + \ldots\right]\mu^m \qquad (k = 1, \ldots, n) \quad (2.2)$$

Computing the coefficients $C_{km}(t)$, as this was done in [1], and keeping hereby all the terms in the expansion of $C_{km}(t)$, we obtain

$$C_{km}(t) = \frac{1}{\Delta_0} \sum_{r=1}^{n} \left[\omega_r \prod_{s \neq r}^{n} (\omega_s^2 - \omega_r^2)\right]^{-1} \int_0^t R_{km}^{(r)}(t') \sin\omega_r (t - t')\, dt' \qquad (2.3)$$

where

$$R_{km}^{(r)}(t) = \sum_{i=1}^{n} \Delta_{ik}(\omega_r^2) H_{im}(t), \qquad H_{im}(t) = \frac{1}{(m-1)!}\left(\frac{d^{m-1} F_i}{d\mu^{m-1}}\right)_{\mu=\beta=\gamma=0} \qquad (2.4)$$

Let us introduce the notation

$$C_m^{(r)}(t) = \left[\Delta_0 \omega_r \prod_{s \neq r}^{n} (\omega_s^2 - \omega_r^2)\right]^{-1} \int_0^t R_{1m}^{(r)}(t') \sin\omega_r (t - t')\, dt' \qquad (2.5)$$

Then, taking into account the relation (1.3), we obtain

$$C_{km}(t) = \sum_{r=1}^{n} p_k^{(r)} C_m^{(r)}(t) \qquad (2.6)$$

The solution of the system (1.1) can thus be expressed in the form

$$x_k(t) = \sum_{r=1}^{n} p_k^{(r)} x^{(r)}(t) \qquad (k = 1, \ldots, n) \qquad (2.7)$$

The quantities $x^{(r)}(t)$ are determined by the formulas

$$x^{(r)}(t) = (A_r + \beta_r)\cos\omega_r t + \frac{B_r + \gamma_r}{\omega_r}\sin\omega_r t + \qquad (2.8)$$

$$+ \sum_{m=1}^{\infty}\left[C_m^{(r)}(t) + \sum_{s=1}^{l} \frac{\partial C_m^{(r)}}{\partial A_s}\beta_s + \sum_{s=2}^{l} \frac{\partial C_m^{(r)}}{\partial B_s}\gamma_s + \ldots\right]\mu^m$$

$$B_1 = 0, \qquad \gamma_1 = 0 \qquad (r = 1, \ldots, l)$$

$$x^{(r)}(t) = \varphi_{r-l}(\beta,\gamma,\mu)\cos\omega_r t + \frac{\psi_{r-l}(\beta,\gamma,\mu)}{\omega_r}\sin\omega_r t + \qquad (r = l+1, \ldots, n) \tag{2.9}$$
$$+ \sum_{m=1}^{\infty}\left[C_m^{(r)}(t) + \sum_{s=1}^{l}\frac{\partial C_m^{(r)}}{\partial A_s}\beta_s + \sum_{s=2}^{l}\frac{\partial C_m^{(r)}}{\partial B_s}\gamma_s + \ldots\right]\mu^m$$

The result obtained can be formulated in the following way.

If the generated solution of the quasi-linear system (1.1) contains l distinct comensurate frequencies which determine a periodic solution with some period T_0, then the corresponding periodic solution of the original quasi-linear system with period $T_0 + \alpha$ (α disappears when $\mu = 0$), which reduces to the generated one when $\mu = 0$, will have the form (2.7) for arbitrary values of l from 1 to n.

Note. There occurred an error in [1]. It was caused by the dropping, in the expansion (2.3), of all the terms with indices from $r = l + 1$ to $r = n$ as terms which might not involve the frequencies $\omega_1, \omega_2, \ldots, \omega_l$, which was a mistake.

Let us consider the integral

$$J_r = \int_0^t R_{km}^{(r)}(t')\sin\omega_r(t-t')\,dt' \qquad (r = l+1, \ldots, n)$$

Suppose that the function $R_{km}^{(r)}$ is a periodic function, with period T_0, whose expansion into a Fourier series is

$$R_{km}^{(r)}(t) = \sum_{n=0}^{\infty}{}' (K_n\cos n\omega_0 t + L_n\sin n\omega_0 t)$$

After some computations we obtain

$$J_r = -\omega_r\sum_{n=0}^{\infty}\frac{K_n\cos n\omega_0 t + L_n\sin n\omega_0 t}{n^2\omega_0^2 - \omega_r^2} +$$
$$+ \omega_r\sum_{n=0}^{\infty}\frac{K_n}{n^2\omega_0^2 - \omega_r^2}\cos\omega_r t + \omega_0\sum_{n=1}^{\infty}\frac{nL_n}{n^2\omega_0^2 - \omega_r^2}\sin\omega_r t$$

The functions $C_{km}^{(r)}(t)$, with $r = l + 1, \ldots, n$, contain in summands a periodic function of period T_0, and the first harmonics with the corresponding frequencies ω_r.

3. Let us consider in greater detail the case of two degrees of freedom when the generated system contains two noncomensurate frequencies. In [3] this case, presented in Section 5, was based on the erroneous results of [1], and, hence, was presented incorrectly.*

We now have the following equations of motion for the system

$$\begin{aligned} a_{11}\ddot{x}_1 + a_{12}\ddot{x}_2 + c_{11}x_1 + c_{12}x_2 &= \mu F_1(x_1, x_2, \dot{x}_1, \dot{x}_2, \mu) \\ a_{21}\ddot{x}_1 + a_{22}\ddot{x}_2 + c_{21}x_1 + c_{22}x_2 &= \mu F_2(x_1, x_2, \dot{x}_2, \dot{x}_2, \mu) \end{aligned} \tag{3.1}$$

Let us look for the periodic solutions of this system with period ω_1. The generated solution in this case becomes

$$x_{10}(t) = A_0 \cos \omega_1 t, \qquad x_{20}(t) = p_1 A_0 \cos \omega_1 t \tag{3.2}$$

Here

$$p_r = p_k^{(r)} = -\frac{c_{11} - \omega_r^2 a_{11}}{c_{12} - \omega_r^2 a_{12}} = -\frac{c_{21} - \omega_r^2 a_{21}}{c_{22} - \omega_r^2 a_{22}} \qquad (r = 1, 2) \tag{3.3}$$

The initial conditions for the system (3.3) will take on the form

$$\begin{aligned} x_1(0) &= A_0 + \beta + \varphi(\beta, \mu), & \dot{x}_1(0) &= \psi(\beta, \mu) \\ x_2(0) &= p_1(A_0 + \beta) + p_2\varphi(\beta, \mu), & \dot{x}_2(0) &= p_2\psi(\beta, \mu) \end{aligned} \tag{3.4}$$

The solution of the system (3.1) can be expressed in the form

$$x_1(t) = x^{(1)}(t) + x^{(2)}(t), \qquad x_2(t) = p_1 x^{(1)}(t) + p_2 x^{(2)}(t) \tag{3.5}$$

The expansion of the functions $x^{(1)}(t)$ and $x^{(2)}(t)$ in powers of the parameters β and μ will be

$$\begin{aligned} x^{(1)}(t) &= (A_0 + \beta)\cos \omega_1 t + \sum_{m=1}^{\infty} \left[C_m^{(1)}(t) + \frac{\partial C_m^{(1)}}{\partial A_0}\beta + \frac{1}{2}\frac{\partial^2 C_m^{(1)}}{\partial A_0^2}\beta^2 + \ldots\right]\mu^m \\ x^{(2)}(t) &= \varphi \cos \omega_2 t + \frac{\psi}{\omega_2}\sin \omega_2 t + \sum_{m=1}^{\infty} \left[C_m^{(2)}(t) + \frac{\partial C_m^{(2)}}{\partial A_0}\beta + \frac{1}{2}\frac{\partial^2 C_m^{(2)}}{\partial A_0^2}\beta^2 + \ldots\right]\mu^m \end{aligned} \tag{3.6}$$

The functions $C_m^{(1)}(t)$ and $C_m^{(2)}(t)$ are determined by the equations

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* In [3] it is necessary to introduce another correction. On page 1671, line 7 from the bottom, the formula $m_1\omega_1 = m_2\omega_2$ must be replaced by

$$m_1 \frac{2\pi}{\omega_1} = m_2 \frac{2\pi}{\omega_2} = T_0,$$

while in line 4 from the bottom, the formula ω_0 must have the form

$$\omega_0 = \frac{\omega_1}{m_1} = \frac{\omega_2}{m_2}$$

$$C_m^{(1)}(t) = \frac{1}{\Delta_0(\omega_2^2 - \omega_1^2)\,\omega_1}\int_0^t R_m^{(1)}(t')\sin\omega_1(t-t')\,dt'$$
$$C_m^{(2)}(t) = \frac{1}{\Delta_0(\omega_1^2 - \omega_2^2)\,\omega_2}\int_0^t R_m^{(2)}(t')\sin\omega_2(t-t')\,dt' \tag{3.7}$$

Here

$$R_m^{(r)}(t) = (c_{22} - \omega_r^2 a_{22})\,H_{1m}(t) - (c_{12} - \omega_r^2 a_{12})\,H_{2m}(t) \qquad (r=1,2) \tag{3.8}$$

The conditions of periodicity for the functions $x^{(1)}(t)$ and $x^{(2)}(t)$ and of their first derivatives are

$$x^{(1)}(T_1+\alpha) = A_0 + \beta, \qquad \dot{x}^{(1)}(T_1+\alpha) = 0$$
$$x^{(2)}(T_1+\alpha) = \varphi(\beta,\mu), \qquad \dot{x}^{(2)}(T_1+\alpha) = \psi(\beta,\mu) \tag{3.9}$$

From these conditions one can find four unknowns α, β, φ and ψ. The problem on the construction of the periodic solutions of the system (3.1) with period T_1, hereby breaks up into two separate problems on the construction of the periodic functions $x^{(1)}(t)$ and $x^{(2)}(t)$ with the same period T_1. These problems can be solved in succession.

The first problem is entirely analogous to the problem of the construction of periodic solutions of a quasi-linear autonomous system with one degree of freedom. In the solution of this problem one determines the quantities α and β. Hereby, the amplitude of the generated solution A_0 is found by means of the equation

$$C_1^{(1)}(T_1) = 0$$

Depending on the multiplicity of the roots of this equation, the quantity β can be represented by means of a series in fractional or integer powers of the parameter μ. The analysis of the possible cases given in [4] can be carried over totally to the consideration of this system.

Let us now proceed to the consideration of the second problem, the construction of the function $x^{(2)}(t)$. For this purpose it is necessary to determine the quantities $\varphi(\beta,\mu)$ and $\psi(\beta,\mu)$. Let us express these quantities in the form of the series

$$\varphi(\beta,\mu) = \sum_{m=1}^{\infty}\left(P_m + \frac{\partial P_m}{\partial A_0}\beta + \frac{1}{2}\frac{\partial^2 P_m}{\partial A_0^2}\beta^2 + \ldots\right)\mu^m$$
$$\psi(\beta,\mu) = \sum_{m=1}^{\infty}\left(Q_m + \frac{\partial Q_m}{\partial A_0}\beta + \frac{1}{2}\frac{\partial^2 Q_m}{\partial A_0^2}\beta^2 + \ldots\right)\mu^m \tag{3.10}$$

Expanding the left-hand sides of the conditions for the periodicity of the function $x^{(2)}(t)$ and its first derivative, into power series of μ and β, and equating the coefficients of like powers of μ in each of the mentioned conditions, we obtain an infinite system of equations for the determination of the coefficients P_m and Q_m:

$$P_m(1-\cos\omega_2 T_1) - \frac{Q_m}{\omega_2}\sin\omega_2 T_1 = W_m(T_1)$$
$$P_m\omega_2\sin\omega_2 T_1 + Q_m(1-\cos\omega_2 T_1) = \dot{W}_m(T_1) \tag{3.11}$$

Let us introduce the notation

$$C_m^{(2)*}(t) = C_m^{(2)}(t) + P_m\cos\omega_2 t + \frac{Q_m}{\omega_2}\sin\omega_2 t \tag{3.12}$$

Then the values of the first three quantities $W_m(T_1)$ can be expressed in the form

$$W_1(T_1) = C_1^{(2)}(T_1)$$
$$W_2(T_1) = C_2^{(2)}(T_1) + N_1\dot{C}_1^{(2)*}(T_1)$$
$$W_3(T_1) = C_3^{(2)}(T_1) + N_1\dot{C}_2^{(2)*}(T_1) + N_2\dot{C}_1^{(2)*}(T_1) + \frac{1}{2}N_1^2\ddot{C}_1^{(2)*}(T_1) \tag{3.13}$$

The quantities N_1, N_2, ... are the coefficients of the expansion of α into a double series in powers of β and μ:

$$\alpha = \sum_{m=1}^{\infty}\left(N_m + \frac{\partial N_m}{\partial A_0}\beta + \frac{1}{2}\frac{\partial^2 N_m}{\partial A_0^2}\beta^2 + \ldots\right)\mu^m \tag{3.14}$$

Furthermore, it should be noted that

$$\dot{W}_m(T_1) = \left(\frac{dW_m(t)}{dt}\right)_{t=T_1}$$

Solving the Equations (3.11), we obtain

$$P_m = \frac{1}{2}W_m(T_1) + \frac{1}{2\omega_2}\frac{\sin\omega_2 T_1}{1-\cos\omega_2 T_1}\dot{W}_m(T_1)$$
$$Q_m = \frac{1}{2}\dot{W}_m(T_1) - \frac{\omega_2}{2}\frac{\sin\omega_2 T_1}{1-\cos\omega_2 T_1}W_m(T_1) \tag{3.15}$$

It follows from these formulas that the quantities P_1 and Q_1, P_2 and Q_2, and so on, can be determined successively. It is not difficult to verify that the functions $C_m^{(2)*}(t)$ are periodic functions of t with period T_1.

Next, we introduce the functions

$$C_{1m}^*(t) = C_m^{(1)}(t) + C_m^{(2)*}(t), \qquad C_{2m}^*(t) = p_1 C_m^{(1)}(t) + p_2 C_m^{(2)*}(t) \tag{3.16}$$

The quantities $H_{im}(t)$, determined by the Formula (2.4) will have the following explicit forms

$$H_{i1}(t) = F_i(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}, 0)$$

$$H_{i2}(t) = \sum_{k=1}^{2} \left(\frac{\partial F_i}{\partial x_k}\right)_0 C_{k1}^* + \sum_{k=1}^{2} \left(\frac{\partial F_i}{\partial \dot{x}_k}\right)_0 \dot{C}_{k1}^* + \left(\frac{\partial F_i}{\partial \mu}\right)_0$$

$$H_{i3}(t) = \frac{1}{2} \sum_{k,\,j=1}^{2} \left(\frac{\partial^2 F_i}{\partial x_k \partial x_j}\right)_0 C_{k1}^* C_{j1}^* + \frac{1}{2} \sum_{k,\,j=1}^{2} \left(\frac{\partial^2 F_i}{\partial \dot{x}_k \partial \dot{x}_j}\right)_0 \dot{C}_{k1}^* \dot{C}_{j1}^* +$$
$$+ \sum_{k,\,j=1}^{2} \left(\frac{\partial^2 F_i}{\partial x_k \partial \dot{x}_j}\right)_0 C_{k1}^* \dot{C}_{j1}^* + \frac{1}{2}\left(\frac{\partial^2 F_i}{\partial \mu^2}\right)_0 + \sum_{k=1}^{2} \left(\frac{\partial^2 F_i}{\partial x_k \partial \mu}\right)_0 C_{k1}^* +$$
$$+ \sum_{k=1}^{2} \left(\frac{\partial^2 F_i}{\partial \dot{x}_k \partial \mu}\right)_0 \dot{C}_{k1}^* + \sum_{k=1}^{2} \left(\frac{\partial F_i}{\partial x_k}\right)_0 C_{k1}^* + \sum_{k=1}^{2} \left(\frac{\partial F_i}{\partial \dot{x}_k}\right)_0 \dot{C}_{k1}^* \qquad (3.17)$$

The subscript 0 of the derivatives of the function F_i denotes that one should substitute x_{10}, x_{20}, $\dot{x}_{10}$, $\dot{x}_{20}$, 0 for the variables x_1, x_2, $\dot{x}_1$, $\dot{x}_2$, μ in these derivatives.

If in the generated system one of the variables can be separated, for example x_1, when $a_{12} = c_{12} = 0$, then one of the coefficients p_r, in our case p_2, becomes infinite. Since the function

$$X^{(2)}(t) = p_2 x^{(2)}(t)$$

and also the quantities

$$\Phi(\beta, \mu) = p_2 \varphi(\beta, \mu), \qquad \Psi(\beta, \mu) = p_2 \psi(\beta, \mu)$$

retain their finite values in this case, the solution (3.1) can be expressed in the form

$$x_1(t) = x^{(1)}(t), \qquad x_2(t) = p_1 x^{(1)}(t) + X^{(2)}(t)$$

The initial conditions now will be

$$x_1(0) = A_0 + \beta, \qquad \dot{x}_1(0) = 0$$
$$x_2(0) = p_1(A_0 + \beta) + \Phi(\beta, \mu), \qquad \dot{x}_2(0) = \Psi(\beta, \mu)$$

The scheme for the computations remains the same except that in place of $x^{(2)}(t)$, $\varphi(\beta, \mu)$, and $\psi(\beta, \mu)$ it is necessary to compute the quantities $X^{(2)}(t)$, $\Phi(\beta, \mu)$, and $\Psi(\beta, \mu)$. If the generated system is reduced to the normal coordinates, then

$$p_1 = 0, \qquad p_2 = \infty$$

In the case when one of the variables is separated in the nonlinear system, it is simpler to solve the problem directly by determining the separated variable.

In conclusion we will derive formulas for $x_1(t)$ and $x_2(t)$ which will represent the solution of the system (3.1). Suppose that β can be determined by means of a power series

$$\beta = \sum_{m=1}^{\infty} A_m \mu^m \tag{3.18}$$

Then the correction α of the period is determined by the series

$$\alpha = T_1 \sum_{m=1}^{\infty} h_m \mu^m \tag{3.19}$$

For the construction of a periodic solution of the system (3.1) with a period that is independent of μ, we make the customary change of the independent variable

$$t = \tau(1 + h_1\mu + h_2\mu^2 + \ldots) \tag{3.20}$$

We will look for a solution that is a function of τ. This solution has the period T_1. The functions $x_1(\tau)$ and $x_2(\tau)$ will be represented by series in integer powers of the parameter μ

$$x_k(\tau) = x_{k0}(\tau) + \mu x_{k1}(\tau) + \mu^2 x_{k2}(\tau) + \ldots \qquad (k = 1, 2) \tag{3.21}$$

whereby

$$x_{1m}(\tau) = x_m^{(1)}(\tau) + x_m^{(2)}(\tau), \qquad x_{2m}(\tau) = p_1 x_m^{(1)}(\tau) + p_2 x_m^{(2)}(\tau) \tag{3.22}$$

The generated solution is given by the Formula (3.2). Hence,

$$x_0^{(1)}(\tau) = A_0 \cos\omega_1\tau, \qquad x_0^{(2)}(\tau) = 0 \tag{3.23}$$

The next two coefficients for both functions will be

$$x_1^{(1)}(\tau) = A_1\cos\omega_1\tau + C_1^{(1)}(\tau) - h_1\tau A_0\,\omega_1\sin\omega_1\tau \qquad x_1^{(2)}(\tau) = C_1^{(2)*}(\tau)$$

$$x_2^{(1)}(\tau) = A_2\cos\omega_1\tau + C_2^{(1)}(\tau) + A_1\frac{\partial C_1^{(1)}}{\partial A_0} + h_1\tau\frac{\partial C_1^{(1)}}{\partial\tau} - (h_1A_1 + h_2A_0)\,\tau\omega_1\sin\omega_1\tau -$$

$$-\frac{1}{2}h_1^2\tau^2A_0\,\omega_1^2\cos\omega_1\tau, \qquad x_2^{(2)}(\tau) = C_2^{(2)*}(\tau) + A_1\frac{\partial C_1^{(2)*}}{\partial A_0} + h_1\tau\frac{\partial C_1^{(2)*}}{\partial\tau} \tag{3.24}$$

In cases when the quantity β is expanded in fractional powers of the parameter μ, the solution $x_k(\tau)$ will also be expanded in terms of the same fractional powers of the parameter μ. The corresponding formulas for the coefficients of the expansion $x_k(\tau)$ can be easily evaluated in a manner analogous to the one used in [4].

4. The presented method of the construction of periodic solutions of autonomous systems with two degrees of freedom can easily be generalized to systems with n degrees of freedom. For example, if the generated system has n distinct frequencies of which l are comensurate with each other, then the problem can be reduced to the problem with l degrees of freedom; after that one can determine the functions $x^{(l+1)}(t), \ldots, x^{(n)}(t)$ successively. In particular, if there exists a frequency, for example ω_1, which is not comensurate with any of the other frequencies, then the construction of the periodic solutions of such a system with period T_1 breaks up into n separate problems for the successive determinations of periodic functions $x^{(1)}(t), \ldots, x^{(n)}(t)$.

The construction of the first one of them, $x^{(1)}(t)$, is entirely analogous to the determination of a periodic solution for a system with one degree of freedom. The construction of the remaining functions, however, is done by the same method, and does not differ from the construction of the function $x^{(2)}(t)$ in the considered case of two degrees of freedom.

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Translated by H.P.T.